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On a Semilinear Wave Equation: The Cauchy Problem and the Asymptotic Behavior of Solutions*

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The semilinear wave equation

$$\Box u + m^2 u + |u|^{p-2} u(V^* |u|^p) = 0$$

in $\Omega = \mathbb{R}^3$, $-\infty < t < \infty$, is studied where $\Box$ denotes the d'Alembertian operator and $*$ means spatial convolution. Under mild assumptions on the real-valued function V and $2 \leqslant p < 3$ the well-posedness of the Cauchy problem is proved. Furthermore, some properties of the solutions of the equation are analyzed such as the asymptotic behavior of local energy as $|t| \to +\infty$ in the case of zero mass. Our results extend that of Perla Menzala and Strauss, where case $p = 2$ was studied.

1. INTRODUCTION

In this paper we shall discuss the semilinear wave equation

$$u_{tt} - \Delta u + m^2 u + |u|^{p-2} u(V * |u|^p) = 0, \tag{1.1}$$

where x runs in $\mathbb{R}^3$, $-\infty < t < \infty$, m^2 is a positive constant, and V is a spatial potential which belongs to $L^{3/5-p}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. In (1.1) the symbol $*$ denotes spatial convolution. We shall consider (1.1) with p in the interval $2 \leqslant p < 3$. The results presented in this paper extend over previous ones obtained in [4] in a joint work with Strauss, where case $p = 2$ was treated.

Instead of (1.1) we could have considered a more general nonlinear term of the form

$$h'(u)(V * h(u)) \qquad \text{with} \quad |h'(u)| \leqslant C\,|u|^{p-1};$$

however, the modifications would be minor, so we prefer to work with (1.1).

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Our basic tools will be the energy method and some interpolation theorems in order to show the well-posedness of the Cauchy problem. In the last section, we use Friedrick's abc method to study the asymptotic behavior of local energy as $|t| \to +\infty$ in the case of zero mass. We remark that some of our results remain valid for $2 \leqslant p < 4$.

2. The Cauchy Problem

In what follows we shall use standard notation. By $L^r(\mathbb{R}^3)$, $1 \leqslant r < \infty$, we shall denote the space of functions in $\mathbb{R}^3$ whose rth powers are integrable with the norm $\|f\|_r^r = \int |f(x)|^r \, dx$ $(1 \leqslant r < \infty)$ and by $L^\infty(\mathbb{R}^3)$ we denote the space of measurable essentially bounded functions in $\mathbb{R}^3$ with the norm $\|f\|_\infty = \text{ess sup}\, |f(x)|$. From now on, an integral sign to which no domain is attached will be understood to be taken over all space $\mathbb{R}^3$. We shall denote by $\operatorname{grad} u$ the gradient of u (in space variables) and $|\operatorname{grad} u|^2 = \sum_{j=1}^3 |\partial u/\partial x_j|^2$. The Laplacian operator is denoted by $\Delta = \sum_{j=1}^3 \partial^2/\partial x^2$. Let K be a positive integer, $H^k(\mathbb{R}^3)$ denotes the Sobolev space of square integrable functions from $\mathbb{R}^3$ into $\mathbb{R}$, which together with their partial derivatives up to order K (in the sense of distributions), belong to $L^2(\mathbb{R}^3)$. The norm in $H^k(\mathbb{R}^3)$ will be denoted by $\| \ \|_{H^k}$. If $u \in H^1(\mathbb{R}^3)$, the Sobolev inequality (in $\mathbb{R}^3$) says that $u \in L^6(\mathbb{R}^3)$ and

$$\|u\|_6 \leqslant C \, \|\operatorname{grad} u\|_2 \tag{2.1}$$

for some positive constant (independent of u) C. From now on, to simplify the notation, we shall denote by C various constants. The symbol $*$ will always denote spatial convolution. If X is a Banach space and I is an interval, we denote by $C(I; X)$ the space of strongly continuous functions from I to X.

Theorem 1 (Local existence). *Let $V\colon \mathbb{R}^3 \to \mathbb{R}$ belong to $L^{3/5-p}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, where $2 \leqslant p < 3$. Let u_0 belong to $H^1(\mathbb{R}^3)$ and $u_1 \in L^2(\mathbb{R}^3)$. Then there exists a maximal interval $I = (T_{\min}, T_{\max})$ with $-\infty \leqslant T_{\min} < 0 < T_{\max} \leqslant +\infty$ and a unique function $u \in C(I; H^1)$, $u_t \in C(I; L^2)$ satisfying Eq. (1.1) in I together with the initial conditions*

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x)$$

for $x \in \mathbb{R}^3$. If $T_{\max} < +\infty$ (or $-\infty < T_{\min}$), then

$$\int (u_t^2 + |\operatorname{grad} u|^2 + m^2 u^2) \, dx \to +\infty$$

as $t \nearrow T_{\max}$ (as $t \searrow T_{\min}$).

Proof. Let us rewrite (1.1) as a system of two equations of first-order in time,

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{bmatrix} 0 & \text{Ident} \\ \Delta - m^2 & 0 \end{bmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ -|u|^{p-2}\, u(V * |u|^p) \end{pmatrix}.$$

By using Segal's theorem [6], we need only show that the map

$$u \overset{N}{\mapsto} -|u|^{p-2}\, u(V * |u|^p)$$

is locally Lipschitz from H^1 into L^2. Let us consider first the case in which $m > 0$. Let us write $V = V_1 + V_2$, where $V_1 \in L^{3/5-p}$ and $V_2 \in L^\infty$. Let $u \in H^1$, then by using the Hölder, Young, and Sobolev inequalities we obtain

$$\begin{aligned} \||u|^{p-2}\, u(V_1 * |u|^p)\|_2 &\leqslant \|u^{p-1}\|_{6/p-1}\, \|V_1 * |u|^p\|_{6/4-p} \\ &\leqslant \|u^{p-1}\|_{6/p-1}\, \|V_1\|_{3/5-p}\, \|u^p\|_{6/p} \\ &= \|u\|_6^{2p-1}\, \|V_1\|_{3/5-p} \\ &\leqslant C\, \|\text{grad}\, u\|_2^{2p-1}\, \|V_1\|_{3/5-p}. \end{aligned} \tag{2.2}$$

Next, let us estimate

$$\begin{aligned} \||u|^{p-2}\, u(V_2 * |u|^p)\|_2 &\leqslant \|u^{p-1}\|_2\, \|V_2 * u^p\|_\infty \\ &\leqslant \|u^{p-1}\|_2\, \|V_2\|_\infty\, \|u^p\|_1 \end{aligned} \tag{2.3}$$

because of the Hölder and Young inequalities. Since $u \in L^2 \cap L^6$, then by interpolation we knwo that

$$\|u\|_{2(p-1)}^{2(p-1)} \leqslant \|u\|_2^{4-p}\, \|u\|_6^{3(p-2)}$$

and

$$\|u\|_p^p \leqslant \|u\|_2^{6-p/2}\, \|u\|_6^{3(p-2)/2}$$

which together with (2.3) give us

$$\begin{aligned} \||u|^{p-2}\, u(V_2 * |u|^p)\|_2 &\leqslant \|u\|_2^{5-p}\, \|u\|_6^{3(p-2)}\, \|V_2\|_\infty \\ &\leqslant C\, \|u\|_2^{5-p}\, \|\text{grad}\, u\|_2^{3(p-2)}\, \|V_2\|_\infty. \end{aligned} \tag{2.4}$$

Thus, (2.2) and (2.4) prove that N takes H^1 into L^2. Now, let us show that N is locally Lipschitz. Let $u, v \in H^1(\mathbb{R}^3)$, and write

$$\begin{aligned} N(u) - N(v) = {} & (|u|^{p-2}\, u - |v|^{p-2}\, v)(V * |u|^p) \\ & + |v|^{p-2}\, v(V * [|u|^p - |v|^p]). \end{aligned}$$

Since $||u|^{p-2}u-|v|^{p-2}v|\leqslant C|u-v|(|u|^{p-2}+|v|^{p-2})$ for some positive constant C, we obtain

$$|N(u)-N(v)|\leqslant C|u-v|(|u|^{p-2}+|v|^{p-2})|V*|u|^p|$$
$$+C|v|^{p-1}(|V|*|u-v|(|u|^{p-1}+|v|^{p-1})). \qquad (2.5)$$

Let us write $V=V_1+V_2$ with $V_1\in L^{3/5-p}$, $V_2\in L^\infty$. Then by using the Hölder and Young inequalities we obtain

$$\begin{aligned}
&\||u-v|\,|u|^{p-2}|V_1*|u|^p|\|_2\\
&\leqslant \||u-v|\,|u|^{p-2}\|_{6/p-1}\|V_1*|u|^p\|_{6/4-p}\\
&\leqslant \||u-v|\,|u|^{p-2}\|_{6/p-1}\|V_1\|_{3/5-p}\|u^p\|_{6/p}\\
&\leqslant \|u-v\|_6\|u^{p-2}\|_{6/p-2}\|V_1\|_{3/5-p}\|u\|_6^p\\
&=\|u-v\|_6\|u\|_6^{2p-2}\|V_1\|_{3/5-p}.
\end{aligned}$$

Similar calculations give us

$$\||u-v|\,|v|^{p-2}|V_1*|u|^p|\|_2\leqslant\|u-v\|_6\|v\|_6^{p-2}\|V_1\|_{3/5-p}\|u\|_6^p,$$
$$\||u-v||u|^{p-2}|V_2*|u|^p|\|_2\leqslant\|u-v\|_6\|u^{p-2}\|_3\|V_2\|_\infty\|u^p\|_2,$$
$$\||v|^{p-1}(V_1*|u-v|\,|u|^{p-1})\|_2\leqslant\|v\|_6^{p-1}\|V_1\|_{3/5-p}\|u-v\|_6\|u\|_6^{p-1},$$
$$\||v|^{p-1}(V_2*|u-v|\,|u|^{p-1})\|_2\leqslant\|v^{p-1}\|_2\|V_2\|_\infty\|u-v\|_2\|u^{p-1}\|_2,$$
$$\||v|^{p-1}(V_1*|u-v|\,|v|^{p-1})\|_2\leqslant\|v\|_6^{2(p-1)}\|V_1\|_{3/5-p}\|u-v\|_6,$$

and

$$\||v|^{p-1}(V_2*|u-v|\,|v|^{p-1})\|_2\leqslant\|v^{p-1}\|_2^2\|V_2\|_\infty\|u-v\|_2.$$

Now (2.5) and all of the inequalities together prove our claim. In case $m=0$ we need only write the equation as $u_{tt}-\Delta u+u=u-|u|^{p-2}u(V*|u|^p)$. Since the map $u\to^N -u+|u|^{p-2}u(V*|u|^p)$ also takes H^1 into L^2 and since it is locally Lipschitz, the proof is finished.

THEOREM 2 (Energy identity). *Let* $V:\mathbb{R}^3\to\mathbb{R}$ *be an even function belonging to* $L^{3/5-p}(\mathbb{R}^3)+L^\infty(\mathbb{R}^3)$ $(2\leqslant p<3)$ *and* $u_0\in H^1$, $u_1\in L^2(\mathbb{R}^3)$, *then the unique solution u obtained in Theorem* 1 *satisfies the energy identity*

$$E(t)=\frac{1}{2}\int(u_t^2+|\operatorname{grad}u|^2+m^2u^2)\,dx$$
$$+\frac{1}{2p}\iint|u(x,t)|^p\,V(x-y)\,|u(y,t)|^p\,dx\,dy=E(0)$$

for any $t\in I$.

Proof. Let us write Eq. (1.1) as

$$u_{tt} - \Delta u + m^2 u = f, \tag{2.6}$$

where $f = -|u|^{p-2} u(V * |u|^p)$. Using linear theory and observing that by Theorem 1, $f \in C(I; L^2)$, we get

$$\frac{1}{2}\frac{d}{dt}\int (u_t^2 + |\operatorname{grad} u|^2 + m^2 u^2)\,dx = \int f u_t\,dx. \tag{2.7}$$

Since V is even it follows that

$$\int f u_t\,dx = -\frac{1}{p}\iint |u(x,t)|^p\, V(x-y)\,\frac{\partial}{\partial t}|u(y,t)|^p\,dx\,dy$$

and

$$\frac{1}{p}\frac{d}{dt}\int |u|^p(V * |u|^p)\,dx = \frac{2}{p}\int |u|^p * \left(V * \frac{\partial}{\partial t}|u|^p\right)dx.$$

Thus, from (2.7) it follows that

$$\frac{1}{2}\frac{d}{dt}\int (u_t^2 + |\operatorname{grad} u|^2 + m^2 u^2)\,dx = -\frac{1}{2p}\frac{d}{dt}\int |u|^p(V * |u|^p)\,dx$$

which proves the theorem.

THEOREM 3 (Global existence). *If* $V: \mathbb{R}^3 \to \mathbb{R}$ *is nonnegative, even function and satisfies all the hypotheses of Theorem* 1, *then* $I = (-\infty, \infty)$.

Proof. First let us suppose that $m > 0$. By Theorem 2 and since V is nonnegative then the energy of u is bounded a priori. Then, $T_{\max}$ (see the last statement of Theorem 1) cannot be finite. In case $m = 0$, we know that $\int (|u_t|^2 + |\operatorname{grad} u|^2)\,dx$ is bounded because V is positive. Therefore, if $T_{\max} < +\infty$, $\int (u_t^2 + |\operatorname{grad} u|^2)\,dx$ would have to approach $+\infty$, which cannot happen because it is bounded. The same idea works for $T_{\min}$.

THEOREM 4. *Let* $V: \mathbb{R}^3 \to \mathbb{R}$ *be a nonnegative, even function which satisfies all of the assumptions of Theorem* 1. *Let* u *and* v *be two solutions as in Theorem* 3. *Then there exists a constant* C, *depending only on the energies of* u *and* v *such that*

$$\|u(\cdot,t) - v(\cdot,t)\|_e \leqslant \|u(\cdot,0) - v(\cdot,0)\|_e \exp(Ct),$$

where

$$\|u(\cdot, t)\|_e^2 = \int (u_t^2 + |\text{grad } u|^2 + m^2 u^2)\, dx.$$

Proof. Take $w = u - v$, then w satisfies the problem

$$w_{tt} - \Delta w + m^2 w + |u|^{p-2} u(V * |u|^p) - |v|^{p-2} v(V * |v|^p) = 0.$$

Thus by using the same idea given in the proof of Theorem 2 we obtain

$$\|u(\cdot, t) - v(\cdot, t)\|_e - \|u(\cdot, 0) - v(\cdot, 0)\|_e$$
$$\leqslant \int_0^t \| |u|^{p-2} u(V * |u|^p) - |v|^{p-2} v(V * |v|^p)\|_2\, ds.$$

Because of the estimates obtained in the last part of the proof of Theorem 1 and use of Theorem 3 we obtain

$$\|u(\cdot, t) - v(\cdot, t)\|_e - \|u(\cdot, 0) - v(\cdot, 0)\|_e$$
$$\leqslant C \int_0^t \|u(\cdot, s) - v(\cdot, s)\|_e\, ds.$$

Now, we use Gronwall's inequality and the proof is complete.

The next theorem assures us that the solution of (1.1) is "smooth" provided that the initial data is "smooth." Since the proof is based on higher order estimates fairly similar to the ones already presented in Theorems 1 and 2, we shall simply sketch the proof and omit the details.

THEOREM 5 (Regularity). *Let V satisfy the conditions of Theorem* 1 *and let K be a nonnegative integer. If $u_0 \in H^{k+1}(\mathbb{R}^3)$ and $u_1 \in H^k(\mathbb{R}^3)$, then $u \in C(I; H^{k+1})$ and $u_t \in C(I; H^k)$.*

Proof. We write (1.1) as a first order system (in t) on the Hilbert space $H^1 + L^2$:

$$\frac{d}{dt}\begin{bmatrix} u \\ u_t \end{bmatrix} = \begin{bmatrix} 0 & \text{Ident} \\ -m^2 + \Delta & 0 \end{bmatrix}\begin{bmatrix} u \\ u_t \end{bmatrix} + \begin{bmatrix} 0 \\ -|u|^{p-2} u(V*|u|^p) \end{bmatrix}$$
$$= A(\phi) + N(\phi),$$

where $\phi = [{u \atop u_t}]$. According to Segal's theorem [6] (see also [5]), we need only verify that the pair $[{u_0 \atop u_1}]$ belongs to the domain of A^k, denoted by $D(A^k)$, and that the nonlinear map N takes $D(A^k)$ into $D(A^k)$ with appropriate bounds. Since $D(A^k) = H^{k+1}(\mathbb{R}^3) + H^k(\mathbb{R}^3)$, then it remains only to show that $N\colon D(A^k) \mapsto D(A^k)$. Let us assume first that $m > 0$. We proceed by induction on K. If $k = 1$, we must estimate the $L^2(\mathbb{R}^3)$ norm of $(\partial/\partial x_j)(|u|^{p-2}\, u(V * |u|^p)$. This is the sum of two terms. The first one is

$$\|(V * |u|^p)(|u|^{p-2}\, u)_{x_j}\|_2 \leqslant (p-1)\, \||u|^{p-2}\, u_{x_j}(V_1 * |u|^p)\|_2 + (p-1)\, \||u|^{p-2}\, u_{x_j}(V_2 * |u|^p)\|_2.$$

By using the Hölder and Young inequality we obtain

$$\begin{aligned} &\||u|^{p-2}\, u_{x_j}(V_2 * |u|^p)\|_2^2 \\ &\quad\leqslant \|V_2\|_\infty^2\, \|u\|_2^{6-p}\, \|u\|_6^{3p-6} \int |u^{p-2} u_{x_j}|^2\, dx \\ &\quad\leqslant \|V_2\|_\infty^2\, \|u\|_2^{6-p}\, \|u\|_6^{5p-10}\, \|u_{x_j}\|_2^{4-p}\, \|u_{x_j}\|_6^{p-2} \\ &\quad\leqslant C\, \|V_2\|_\infty^2\, \|u\|_2^{6-p}\, \|\operatorname{grad} u\|_2^{5p-10}\, \|u_{x_j}\|_2^{4-p}\, \|\Delta u\|_2^{p-2} \end{aligned} \tag{2.8}$$

because of the Sobolev inequality. Similarly, we obtain the inequalities

$$\begin{aligned} ||u|^{p-2}\, u_{x_j}(V_1 * |u|^p)\|_2 &\leqslant \|V_1 * |u|^p\|_{6/4-p}\, \|u\|_6^{p-2}\, \|u_{x_j}\|_6 \\ &\leqslant \|V_1\|_{3/5-p}\, \|u\|_6^{2p-2}\, \|u_{x_j}\|_6 \\ &\leqslant C\, \|V_1\|_{3/5-p}\, \|\operatorname{grad} u\|_2^{2p-2}\, \|\Delta u\|_2. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9) it follows that

$$\|(V * |u|^p)(|u|^{p-2}\, u)_{x_j}\|_2 \leqslant C(V)\, \|u\|_{H^2}^{2p-1}. \tag{2.10}$$

The second term that we have to estimate is

$$\||u|^{p-2}\, u(V * |u|^{p-1}\, u_{x_j}\|_2.$$

Since

$$\begin{aligned} \||u|^{p-2}\, u(V_2 * |u|^{p-1}\, u_{x_j})\|_2 &\leqslant \|V_2\|_\infty\, \|u^{p-1}\|_2\, \|u^{p-1} u_{x_j}\|_2 \\ &\leqslant \|V_2\|_\infty\, \|u\|^{4-p/2}\, \|u\|^{3(p-2)/2}\, \|u^{p-1} u_{x_j}\|_2 \end{aligned}$$

by interpolation, now we use Hölder's inequality to obtain

$$\leqslant \|V_2\|_\infty\, \|u\|_2^{4-p/2}\, \|u\|_6^{5p-8/2}\, \|u_{x_j}\|_2^{3-p/2}\, \|u_{x_j}\|_6^{p-1/2}.$$

Because of the Sobolev inequality we get

$$\begin{aligned} &\||u|^{p-2}\, u(V_2 * |u|^{p-1}\, u_{x_j}\|_2 \\ &\quad\leqslant C\, \|V_2\|_\infty\, \|u\|_2^{4-p/2}\, \|\operatorname{grad} u\|_2^{5p-8/2}\, \|u_{x_j}\|_2^{3-p/2}\, \|\Delta u\|_2^{p-1/2} \\ &\quad\leqslant C(V_2)\, \|u\|_{H^2}^{2p-1}. \end{aligned} \tag{2.11}$$

It remains to estimate the term

$$
\begin{aligned}
\| |u|^{p-2} u(V_1 * |u|^{p-1} u_{x_j})\|_2 &\leqslant \| V_1 \|_{3/5-p} \| u^{p-1} u_{x_j} \|_{6/p} \| u \|_6^{p-1} \\
&\leqslant \| V_1 \|_{3/5-p} \| u \|_6^{2p-2} \| u_{x_j} \|_6 \\
&\leqslant C \| V_1 \|_{3/5-p} \| \operatorname{grad} u \|_2^{2p-2} \| \Delta u \|_2 \\
&\leqslant C(V_1) \| u \|_{H^2}^{2p-1}. \qquad (2.12)
\end{aligned}
$$

From (2.10)–(2.12) we conclude that $\|N(\phi)\| \leqslant C(\|\phi\|) \|\phi\|$. It is also clear from the estimates that $N(\phi) - N(\psi)$ can be similarly treated to estimate the $L^2(\mathbb{R}^3)$ norm of $(\partial/\partial x_j)(|u|^{p-2} u(V * |u|^p) - |v|^{p-2} v(V * |v|^p))$ in terms of $\|u\|_{H^2}$, $\|v\|_{H^2}$, and $\|u - v\|_{H^2}$ to obtain $\|N(\phi) - N(\psi)\| \leqslant C(\|\phi\|, \|\psi\|) \|\phi - \psi\|$. This proves the case $k = 1$. For $k > 1$, we proceed by induction by estimating the higher order derivatives. We shall omit the details.

Remark 1. By using the same procedure as in the case $p = 2$ (see [4]) we can show the following property of the solutions of (1.1): Let us assume that V satisfies all of the assumptions of Theorem 1. Let $u_0 \in H^1(\mathbb{R}^3)$ and $u_1 \in L^2(\mathbb{R}^3)$, then the values of the solution u (of (1.1)) in $D_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^3 \times \mathbb{R}, |x - x_0| < r, t = t_0\}$ depend only on the values of the initial data in $\{(x, 0), |x - x_0| < r + |t_0|\}$, provided that t_0 belongs to the interval of existence and $r > 0$.

3. Local Energy Decay in the Case $m = 0$

In this section we shall study the asymptotic behavior of local energy in the case in which $m = 0$. We use Friedrick's abc method.

Lemma 1. *Let $m = 0$ and $V: \mathbb{R}^3 \to \mathbb{R}$ be an even function, nonnegative, and belonging to $L^{3/5-p}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. Let us assume that V is a C^1 function. If $u(x, t)$ is a smooth solution of* (1.1) *with initial data of compact support, then*

$$
\begin{aligned}
(1) \quad & \frac{d}{dt} \int [te(u) + (ru_r + u) u_t] \, dx \\
& \qquad - \frac{1}{2p} \int ([rV_r + (7 - 2p) V] * |u|^p) |u|^p \, dx = 0
\end{aligned}
$$

and

$$(2)\quad \frac{d}{dt}\int \left[a(x,t)+\frac{1}{p}r^2\,|u|^p(V*|u|^p)\right]dx$$

$$-\frac{t}{2}\int |u|^p\,([(7-2p)\,V+x\cdot\nabla V]*|u|^p)\,dx$$

$$-\frac{(7-2p)}{p}\int r^2\,|u|^p(V*|u|^{p-1}\,u_t)\,dx=0,$$

where $\partial/\partial r = x/r\cdot\mathrm{grad}$, $e(u)=\frac{1}{2}u_t^2+\frac{1}{2}|\mathrm{grad}\,u|^2+(1/2p)(V*|u|^p)\,|u|^p$ *and* $a=(t^2+r^2)\,e(u)+2tuu_t+2tru_ru_t-u^2$, $r=|x|$.

Proof. (1) We use the multiplier

$$M(u)=tu_t+ru_r+u$$

to get

$$0=[u_{tt}-\Delta u+|u|^{p-2}\,u(V*|u|^p)]\,M(u)=\frac{\partial}{\partial t}X+\mathrm{div}\,Y+Z, \tag{3.1}$$

where

$$X=te(u)+(ru_r+u)\,u_t,$$

$$Y=-\,tu_t\,\mathrm{grad}\,u-\frac{x}{2}u_t^2+\frac{x}{2}|\mathrm{grad}\,u|^2-ru_r\,\mathrm{grad}\,u$$

$$-\,u\,\mathrm{grad}\,u+\frac{x}{p}|u|^p\,(V*|u|^p)$$

and

$$Z=\left(\frac{2p-7}{2p}\right)|u|^p(V*|u|^p)-\frac{1}{p}\sum_{j=1}^{3}x_j\left(\frac{\partial v}{\partial x_j}*|u|^p\right)|u|^p$$

$$-\frac{t}{2p}|u|^p\,(V*|u|^p)_t+\frac{t}{2}|u|^{p-1}\,u_t(V*|u|^p).$$

Since V is even we know that

$$\frac{1}{p}\int |u|^p\,(V*|u|^p)_t\,dx=\int |u|^{p-1}\,u_t(V*|u|^p)\,dx$$

and since $\partial v/\partial x_j$ is odd we obtain

$$\int x_j\left(\frac{\partial v}{\partial x_j}*|u|^p\right)|u|^p\,dx=\frac{1}{2}\int\left(x_j\frac{\partial v}{\partial x_j}*|u|^p\right)|u|^p\,dx.$$

By integrating (3.1) (in x) and using the above observations we get (1).

To show (2): We use the multiplier

$$M(u) = (t^2 + r^2)\, u_t + 2rtu_r + 2tu$$

and proceed as above.

THEOREM 6. *Under the same assumptions of Lemma* 1 *let us suppose that* $|x|^{7-2p} V(x)$ *is nonincreasing in* $r = |x|$, *then, for any* $0 < \varepsilon < 1$

$$\lim_{|t| \to +\infty} \int_{|x| < (1-\varepsilon)|t|} (u_t^2 + |\operatorname{grad} u|^2)\, dx = 0.$$

Proof. Our assumptions on V imply that $rV_r + (7-2p)\, V \leqslant 0$. Therefore from Lemma 1 (1) we obtain

$$\frac{d}{dt} \int [te(u) + (ru_r + u)\, u_t]\, dx \leqslant 0. \tag{3.2}$$

Integrating (3.2) from zero to $T > 0$ (any $T > 0$) we obtain

$$\int [te(u) + (ru_r + u)\, u_t|_{t=T}\, dx \leqslant C, \tag{3.3}$$

where C is a positive constant which depends only on the initial data (at $t = 0$). Let us denote

$$\mathbf{F}(x, t) = \operatorname{grad} u(x, t) + (x/r^2)\, u(x, t).$$

A direct calculation shows that

$$|\operatorname{grad} u|^2 = |\mathbf{F}|^2 - \operatorname{div}\left[\frac{x}{r^2} u^2\right]. \tag{3.4}$$

By substituting (3.4) in (3.3) we obtain

$$\begin{aligned} &\frac{T}{2} \int \left[u_t^2(x, T) + |\mathbf{F}(x, T)|^2 - \operatorname{div}\left(\frac{x}{r^2} u^2(x, T)\right)\right] dx \\ &+ \frac{T}{2p} \int |u(x, T)|^p (V * |u|^p)\, dx \\ &+ \int (ru_r(x, T) + u(x, T))\, u_t(x, T)\, dx \leqslant C. \end{aligned} \tag{3.5}$$

Now let us fix $R > 0$ such that the support of u_0 and u_1 is contained in the ball $B_R = \{x, |x| < R\}$. Then by Remark 1 it follows that $u(x, T)$ has support in $\{|x| < T+R\}$. Observe that $x \cdot Fu_t = ru_r u_t + uu_t$. Therefore, let us rewrite (3.5) as

$$\int \left[\frac{(T+R)}{2}(u_t^2(x,T) + |\mathbf{F}(x,T)|^2) + x \cdot F(x,T)\, u_t(x,T)\right] dx$$
$$+ \frac{T}{2p}\int |u(x,T)|^p (V * |u|^p)\, dx$$
$$- \frac{R}{2}\int (u_t^2(x,T) + |\mathbf{F}(x,T)|^2)\, dx \leqslant C. \tag{3.6}$$

Observe that $|x \cdot Fu_t| \leqslant ((T+R)/2)(|\mathbf{F}|^2 + |u_t|^2)$ if $|x| \leqslant R+T$. Therefore the integrand of the first integral of the left-hand side of (3.6) is positive. Thus we can get a lower bound of it by throwing away the part over $\{|x| > (1-\varepsilon)T\}$. Thus from (3.6) we obtain

$$\frac{\varepsilon}{2}(T+R)\int_{|x|<(1-\varepsilon)T} (u_t^2 + |\text{grad}\, u|^2)\, dx + \frac{T}{2p}\int |u(x,T)|^p (V * |u|^p)\, dx$$
$$- \frac{R}{2}\int (u_t^2(x,T) + |\mathbf{F}(x,T)|^2)\, dx \leqslant C. \tag{3.7}$$

By energy conservation we know that the term

$$\frac{R}{2}\int (u_t^2(x,T) + |\mathbf{F}(x,T)|^2)\, dx$$

is bounded. Therefore from (3.7) we obtain

$$\int_{|x|<(1-\varepsilon)T} (u_t^2 + |\text{grad}\, u|^2)\, dx = O(T^{-1})$$

which proves the theorem.

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